9 - Propositional logic

A **proposition** is a declarative sentence that is always true or always false.

Not propositions:

"I can haz cheezeburger?" (Question) "Pizza tastes good!" (Opinion)

"Fly, you fools!" (Command) "Sum of two squares." (Not even a sentence)

Propositions:

"3+x=5 if x=2." (True proposition) "4 is prime." (False proposition)

"3+x=5 if x=0." (False proposition) "There are 7 days in a week." (True proposition)

Logical connectives "and", "or", and "not" connect propositions p and q:

¬p ("not p", "negation of p", "it is not the case that p") is true <-> p = False.

p∧q ("p and q", "p but q", "conjunction of p and q") is true <-> p=True and q=True.

p∨q ("p or q", "disjunction of p and q") is true <-> at least one of p and q is true.

p⊕q ("p xor q", "p exclusive-or q") is true <-> exactly one of p and q is true.

Example 1. Let p = "It is hot" and q = "It is sunny". Then:

¬p = "It is not hot." "It is sunny but not hot" = q∧¬p

p∧q = "It is hot and sunny." "It's neither hot nor sunny" = ¬p∧¬q

Example 2. (Olivia Rodrigo) "Good for you, you look happy and healthy, not me"

"You look happy" ∧ "You look healthy" ∧ ¬"I look happy" ∧ ¬"I look healthy"

Example 3.

"You can phone or email me." (inclusive or)

"You can have soup or salad as a side." (exclusive or)

Optional Homework due March 25th or 26th.
Show your work. Answer without work receives no credit.

1. Write the statements (a)-(e) in terms of the letters h = "Jon is healthy", w = "Jon is wealthy", s = "Jon is wise", and the symbols ¬, ∧, ∨, ⊕.
(a) Jon is healthy and wealthy but not wise. (b) Jon is not wealthy but he is healthy and wise.
(c) Jon is neither healthy, wealthy, nor wise. (d) Jon is neither wealthy nor wise, but he is healthy.
(e) Jon is wealthy, but he is not both healthy and wise.

2. Write the statements (a)-(c) in terms of the letters p = "x > 5", q = "x = 5", r = "10 > x", and the symbols ¬, ∧, ∨, ⊕.
(a) x ≥ 5 (b) 10 > x > 5 (c) 10 > x ≥ 5

3. True or false:
(a) "Either 12321 ≡ 0 (mod 3) or 12321 ≡ 0 (mod 9)".
(b) "12321 ≡ 0 (mod 3)" ⊕ "12321 ≡ 0 (mod 9)"

A **truth assignment** is an assignment of True (T) or False (F) to a propositional variable. A **truth table** lists all such assignments and the truth value of the proposition. Two propositions are **logically equivalent**, written $p \equiv q$, if their truth values are the same under every truth assignment.

p	¬p
T	F
F	T

p	q	p∧q
T	T	T
T	F	F
F	T	F
F	F	F

p	q	p∨q
T	T	T
T	F	T
F	T	T
F	F	F

p	q	p⊕q
T	T	F
T	F	T
F	T	T
F	F	F

Example 4. Are "It is neither hot nor sunny" and "It is not (either hot or sunny)" logically equivalent?

We interpreted "It is neither hot nor sunny" as "It is not hot and not sunny", so we want to know if "It is not hot and not sunny" and "It is not (either hot or sunny)" the same.

If we let p = "It is hot" and q = "It is sunny", then we want to know if:

$$\neg p \wedge \neg q \stackrel{?}{\equiv} \neg(p \vee q)$$

We show these two are the same by comparing their truth tables.

p	q	¬p	¬q	¬p∧¬q
T	T	F	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

p	q	p∨q	¬(p∨q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

The columns for ¬p∧¬q and ¬(p∨q) have the same truth values, so ¬p∧¬q ≡¬(p∨q).

So "It is neither hot nor sunny" can both be interpreted as "It is not hot and not sunny" and "It is not (either hot or sunny)".

The result $\neg(p \vee q) \equiv \neg p \wedge \neg q$ is a part of de Morgan's Law for logic.

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4. Show $\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$ (2nd half of de Morgan's law)

5. Show $(p \wedge q) \vee r \not\equiv p \wedge (q \vee r)$